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Sigma One

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ABSTRACT

We demonstrate that it is possible to calculate not only the mean of an underlying population but also its dispersion, given only a single observation and physically reasonable constraints (i.e., that the quantities under consideration are non-negative and bounded). We suggest that this counter-intuitive conclusion is in fact at the heart of most modeling of astronomical data.

“Probability theory is nothing but common sense reduced to calculation.”

P.S. Laplace (1819) as quoted by E.T. Jaynes (2003)

1. INTRODUCTION

In their discussion of optimizing search strategies for the simultaneous discovery and characterization of variable stars using predetermined but highly non-uniform sampling, Madore & Freedman (2005) investigated the small-sample limits. Their emphasis was on situations where only a handful of observations (2-20 say) might be available from which it would be necessary to determine periods, measure mean magnitudes and derive full amplitudes. In this paper we discuss how it is possible use a single observation to derive confidence intervals on the mean (i.e., limits on variability, or estimates of the population variance, or its full amplitude, etc.).

Shortly after the original manuscript was completed it was pointed out that similar arguments have been made much earlier by Gott (1993). Gott was interested in the estimation of ages and lifetimes of events based single instances; and he presented confidence intervals for the longevity of an object given a single observation of its present age. He assumed a flat prior (of finite duration), and took the stance that there cannot be anything special about

any given time that a random observation is made of an object that has a finite lifetime. We return to a contextual discussion of this example, and a generalization of it to modeling, at the end of this paper.

2. ONE OBSERVATION

What information can be derived about the parent population from a single, isolated (non-negative) observation? Here we are clearly working at the observational information limit, and so a number of plausible assumptions (“priors”) will of necessity enter the calculations. A tacit assumption is that the observation in question is drawn from an underlying population having some dispersion¹ that we are trying to estimate. We also assume that this is a physical object and that all values (observed or considered in the calculation) are therefore non-negative.

Lacking any information to the contrary we take the single observed data point y_1 to be the best available estimate of the true mean, $\langle Y \rangle$, thus $\langle Y \rangle = y_1$. It follows directly from the non-negativity assumption that the maximum possible semi-amplitude of Y is $A/2 = (\langle Y \rangle - 0.0) = y_1$. Invoking a symmetric prior would give a full amplitude $A = 2 \times y_1$. Having given the flavor of this argument in the above few lines we now more closely investigate a variety of priors, but first we introduce a new tool, the “*g-factor*”.

We ask the following question: if random samples, y_i , are drawn from the parent distribution and considered one at a time, what is the auxiliary distribution of multipliers, “*g-factors*”, that would convert the observation itself into its own “error”, (ϵ_i) , where ϵ_i is defined as being the absolute value of the distance of the data point from the true population mean (i.e., $\epsilon_i \equiv |\langle Y \rangle - y_i|$). By this definition $g_i \times y_i = \epsilon_i$ and so $g_i = |\langle Y \rangle / y_i - 1.0|$. With the *g-factor* distribution in hand, for any given prior distribution, it is then possible to calculate specific multipliers ($g(50.0)$, $g(68.5)$, $g(90.0)$, *etc.*, by integrating over the *g-factor* distribution function) that will guarantee that set fractions (50.0%, 68.5%, 90.0%, *etc.*) of the dataset will fall within the range $y_1 \pm g(50.0) \times y_1$, $y_1 \pm g(68.5) \times y_1$, *etc.*, where y_1 is again our estimate for $\langle Y \rangle$.

¹Of course, if we assumed that the underlying population had no dispersion our task would already be completed. To first order most observers make this default assumption, modulo observational error.

3. PRIORS

We now consider five realizations of four plausible underlying population distribution functions – a Poisson distribution, a uniform distribution, an exponential, and two Gaussians.

3.1. Poisson Distribution

In this first example, invoking the Poisson distribution as being the appropriate form for the underlying parent distribution, it is almost trivially simple to demonstrate that a single observation can deliver both an average and a meaningful value for the dispersion of the underlying distribution. We first recall that the Poisson distribution for rare events has the form $P(y) = \lambda^y e^{-\lambda} / y!$ where it is well known that the distribution is determined by a single variable λ , and that λ is numerically equal to both the mean and the variance of this distribution. Therefore, if we have a single observation y_1 , which we assume comes from a Poisson process then, as a first approximation we can equate y_1 with the mean value for that population $\langle Y \rangle \equiv \lambda = y_1$, and by invoking the known (definitional) equality of the mean and the variance for a Poisson process, we have therefore also measured the variance², with only a single observation.

3.2. Uniform Parent Population

We now consider a uniform distribution spanning the non-negative interval $[0,2]$ (Figure 1, upper panel). The distribution of *g-factors* is expected to be highly skewed: points near zero will require very large factors to make their error bars overlap with the true mean; however, all points greater than the mean (that is, fully half of the ensemble for a uniform prior) will have calculated error bars that overlap the mean for multiplicative *g-factors* that are all less than 0.5.

We first solved for the *g-factor* distribution function by simulation. We successively drew 100,000 observations from a uniform distribution, calculated the factor that needed to be applied to that number such that $y_1 \pm g_u \times y_1$ just overlaps the true mean $\langle Y_u \rangle$. In Figure 1 the results of that computer simulation are shown, where the distribution function of multiplicative *g-factors* is found in the lower panel, and the parent distribution of individual

²And while not putting to fine a point to it, we also note that this single observation also determines several higher moments including the *skew* $= \lambda^{-1/2}$ and the *kurtosis* $= \lambda^{-1}$

observations going into the simulation is given in the upper panel. The lower panel also shows where the 50, 68.5, and 90% confidence intervals are found for the *g-factor* distribution corresponding to this uniform prior.

If the underlying population, from which a single data point y_1 is drawn, is itself uniformly populated and non-negative then $y_1 \pm 0.41 \times y_1$ will contain the mean of the parent population 50% of the time. Other selected confidence intervals are given in Table 1.

After the computer simulations were completed it proved possible for us to derive a closed-form analytic solution for multiplicative factors, g_u as a function of the confidence intervals (CI) associated with this case of a uniform prior. We give these solutions below; they are based on a simple mapping of the uniform distribution into its *g-factors* (using the absolute values of the *g-factors* accounts for the curious shape of the distribution in the lower panel of Figure 1). The confidence intervals are then found by integrating the normalized functional form of that mapping up to required value of CI .

$$g_u = (\sqrt{1.0 + 4(CI)^2} - 1.0)/(2CI) \quad [CI \leq 2/3]$$

$$g_u = (1.0 - 2CI)/(2(CI - 1.0)) \quad [CI \geq 2/3]$$

Exemplary values for common confidence intervals for the uniform prior are given in the second column of Table 1. For the uniform prior they were calculated from the analytic solution and are confirmed by the simulations; all other solutions for other priors were derived from the computer simulations.

3.3. Exponential Parent Population

Another plausible parent population is the exponential distribution. Here we investigate the *g-factors* corresponding to that distribution function. We have chosen to simulate an exponential distribution with mean $\langle Y_e \rangle = 1.0$. The results of the inversion are shown in Figure 2 and the third column of Table 1. The contrast between this and the uniform prior is clear. Because significantly more samples will come with values close to zero their contribution to the distribution of multiplicative factors will increase. This forces the individual *g-factors* for specific confidence intervals to higher values as compared to the uniform prior. Inspection of Table 1 confirms this quantitatively for all confidence intervals listed.

3.4. Two Gaussian Distributions

Since the uniform and the exponential priors are characterized by a single parameter, the mean, they are far more easily constrained by the observation(s). But, for completeness and for illustrative purposes we consider here two Gaussian distributions each with a mean of unity, but having differing dispersions. We make no claim that these two parameters can be constrained by a single observation. We simply want to illustrate quantitatively that the uniform prior at least (and the exponential prior in its extreme) both encompass the results for these Gaussian priors as well. That is, by adopting the *g-factors* for a uniform prior one will also encompass the confidence intervals derived for a variety of Gaussian distributions have the same (unit) mean and certainly any dispersion less than 0.5.

The results for these additional priors are given graphically in Figures 3 and 4, and in Columns 5 and 6 of Table 1.

4. Discussion and Conclusions

Our interest in this formalism started with a desire to characterize luminosity variations of astronomical objects (in the first instance through their first two moments, chosen to be their means and amplitudes) having a highly restricted number of observations. Here we have taken that thought experiment to its limit of a singular observation. For an underlying distribution of finite extent (or duration) the dispersion and the full amplitude are equivalent, differing only by a constant scale factor in any given case. Accordingly, our derivation of the variance of a population based on a single observation is conceptually equivalent to Gott’s (1993) derivation of future longevity (that is, Gott’s longevity is a one-directional semi-amplitude) for an object based on a “single” observation of its present age.

When dealing with physical observations certain assumptions are tacitly taken for granted. The obvious assumptions are, that those quantities are non-negative, and that the underlying population from which they are drawn does not have an infinite range (in time, mass, energy, size, *etc.*) However, those same assumptions carry additionally useful (prior) quantitative information that can be used to constrain limits on observations as they are obtained. This paper had the intent of making those assumptions explicit and then formalizing the mathematical consequences expressed as confidence intervals on the underlying population as derived from one observation.

While it was our hope that this intentionally short contribution might stimulate others into finding applications not obvious to the author, the referee requested that examples be given of how this formalism might be applied to astronomy. In keeping with the spirit of

this paper we offer a modest example from the past, and predict that there will be future examples; all of this based on a sample of $N = 1$.

Laplace developed his “rule of succession” when confronted with a question as to the mathematical (not the physical) probability that the Sun will rise tomorrow given its past (statistical) performance. After observing N events, Laplace derived that the probability of the next occurrence was $(N + 1)/(N + 2)$. This would suggest that on the first day ($N = 0$) the probability of the Sun rising was actually 50:50. On the second day ($N = 1$) the probability would have gone up to 66%, and so on. Gott (1993) asked a similar question, not just about the next occurrence of something that has a past persistence, but about the sum of all future occurrences. How long will a thing last, given that we know how old it is now? Gott was reformulating Laplace’s question to be, if we have a single measurement ($N = 1$) of the age of something, what can one say about its total lifetime (its full amplitude), or rather its future longevity (a semi-amplitude). Of course, any such prediction is best described in terms of probabilities, and so rather than predicting a firm lifetime based on a precise age, Gott predicted a forward-looking probability distribution (expressed as a variance) based on a backward-looking age.

One could argue that Gott actually required two observations: one of the time at which something began and another of the time at which the prediction was being made. This may be seen as quibbling but it is, in fact, equivalent to our physical prior, state above, that none of the quantities to which this method applies can drop below zero or become infinite in amplitude (mass, luminosity, time, etc).

Grounded with post-dictions on the longevity of the Soviet Union and the Berlin Wall, Gott went on to predict the longevity of a variety of things astronomically big and small: from the expected demise of *Nature* magazine itself (somewhere between 3.15 and 4,800 years), to the probability that we will end as a civilization (in 5,100 to 8 million years with 95% probability), or colonize the Galaxy (the odds are against it). Each of these predictions were based on a single observation, $N = 1$ and a uniform prior.

It is quite clear that the uniform, non-negative prior distribution of data points discussed above is at one extreme (of simplicity, or of ignorance.) However, this extreme is also rather inclusive. If the true range of the underlying distribution is smaller, more centrally peaked, or more skewed toward the upper bound of the distribution function, than a uniform distribution, then their confidence intervals will also be smaller than those calculated for a uniform prior; under those conditions the uniform prior is likely to provide a conservative *upper* bound on the uncertainty.

4.1. Is This Just Another Name for Modeling?

Finally, we suggest that aspects of the scientific enterprise as a whole, as practiced by many astronomers in interpreting observational data, might simply be a generalization of the the Sigma One methodology discussed here. Seen in that retrospective light, Sigma One becomes a fairly benign and low-level form of what would otherwise be called “modeling”.

Consider the observation of a color-magnitude diagram for a composite stellar population, in a nearby galaxy, say. Based on that single observation one could ask what the magnitude and color of any given star might be on the next exposure (whenever that may be.) Depending on the amount of prior knowledge about the underlying distribution function for that star one could make a prediction. Indeed we do this all the time. It is known that intrinsic variables (Cepheids, Miras, RR Lyrae stars, etc.) occupy fairly well-delineated regions of the color-magnitude diagram. Armed with known amplitudes and timescales one could invoke those distribution functions with their specific means and variances to predict the expected variance in those selected stars. Stars in regions not known to be variable on those same time-scales would have different priors used to predict their means and variances.

But all of this could also be recast into a very different form of the underlying distribution function, in the case where extremely long (astronomically long) timescales are being considered. The predictive prior for a single observation would then become stellar evolution theory itself. That is, given a star observed (once) today at a given place in the color-magnitude diagram, what is its color and magnitude distribution function integrated over its projected future existence? And then how might the ensemble change with time? We apparently have no problem in undertaking population synthesis modeling, for example, taking a single integrated spectrum and/or a single color-magnitude diagram and extrapolating it to encompass the entire life history (backward and forward in time) of a given star and/or its associated contemporary population (an entire galaxy). So our point here is that if we are comfortable extracting very complex “moments” (in time and composition, etc.) from single (but admittedly very rich) observations by invoking very complicated priors (i.e., models), then it should come as no surprise that it is possible to extract more than just one moment (i.e., a mean and a variance at least) from a single data point by assuming very simple priors (i.e., models), in the form of well known, commonly invoked, but simple, distribution functions.

I thank Wendy Freedman for pointing out both the *prior* nature and the *a posteriori* relevance of the Gott (1993) Nature paper. Discussions with David Hogg were both stimulating and illuminating, as always. And finally, I would like to thank both the Editor, Jay Gallagher for his patience in dealing with this paper as it slowly evolved, and the

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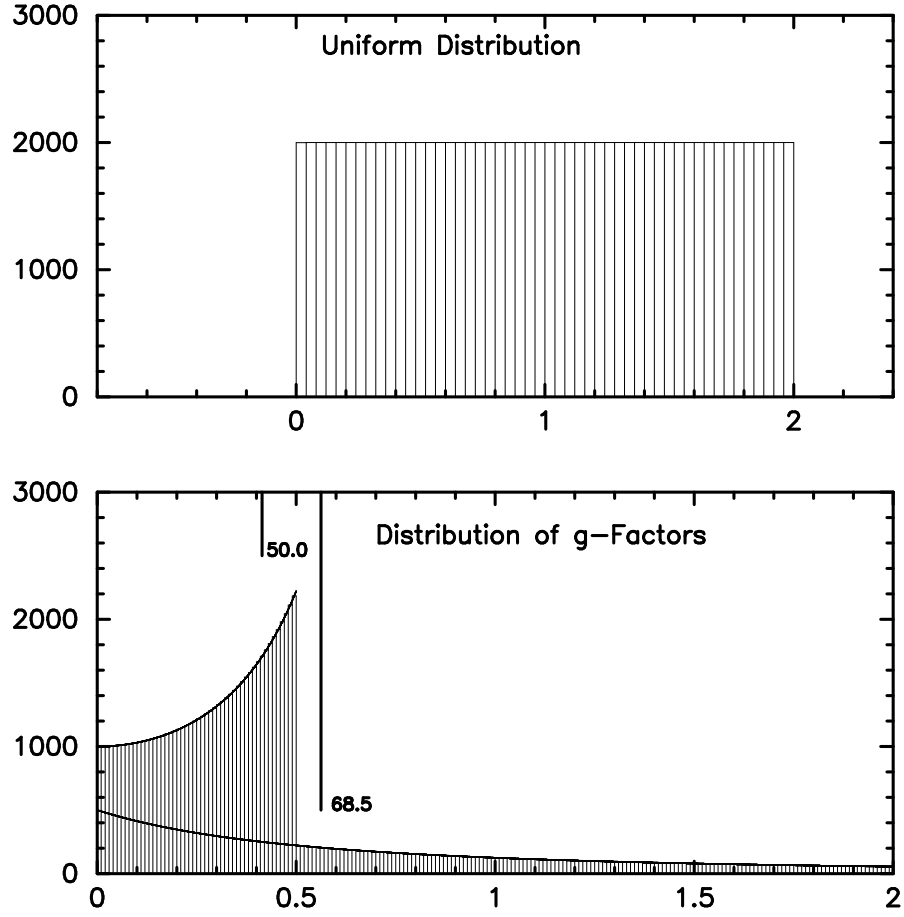


Fig. 1.— – Uniform Prior. The upper panel shows the input distribution of points having a uniform distribution with a mean of 1 and a width of 2. The lower panel shows the cumulative distribution of the multiplicative g -factors (as a function of the g -factor itself) needed to convert random samples taken from the upper panels into the mean. The g -factors required to give 50% and 68.5% (one sigma) confidence intervals for this underlying (uniformly distributed) sample are given as vertical bars and labeled accordingly

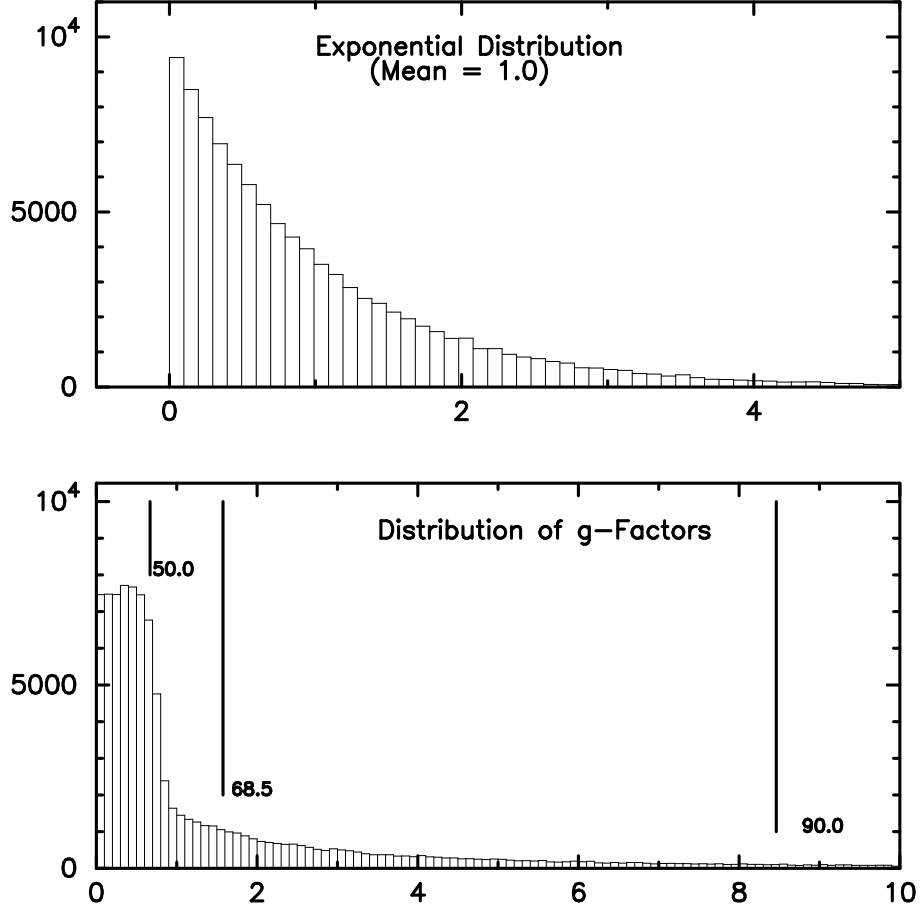


Fig. 2.— – Exponential Prior. The upper panel shows the input distribution of points having an exponential distribution with a mean of 1. The lower panel shows the cumulative distribution of the multiplicative *g-factors* (as a function of the *g-factor* itself) needed to convert random samples taken from the upper panels into the mean. The *g-factors* required to give 50%, 68.5% (one sigma) and 90.0% confidence intervals for this underlying (exponentially distributed) sample are given as vertical bars and labeled accordingly

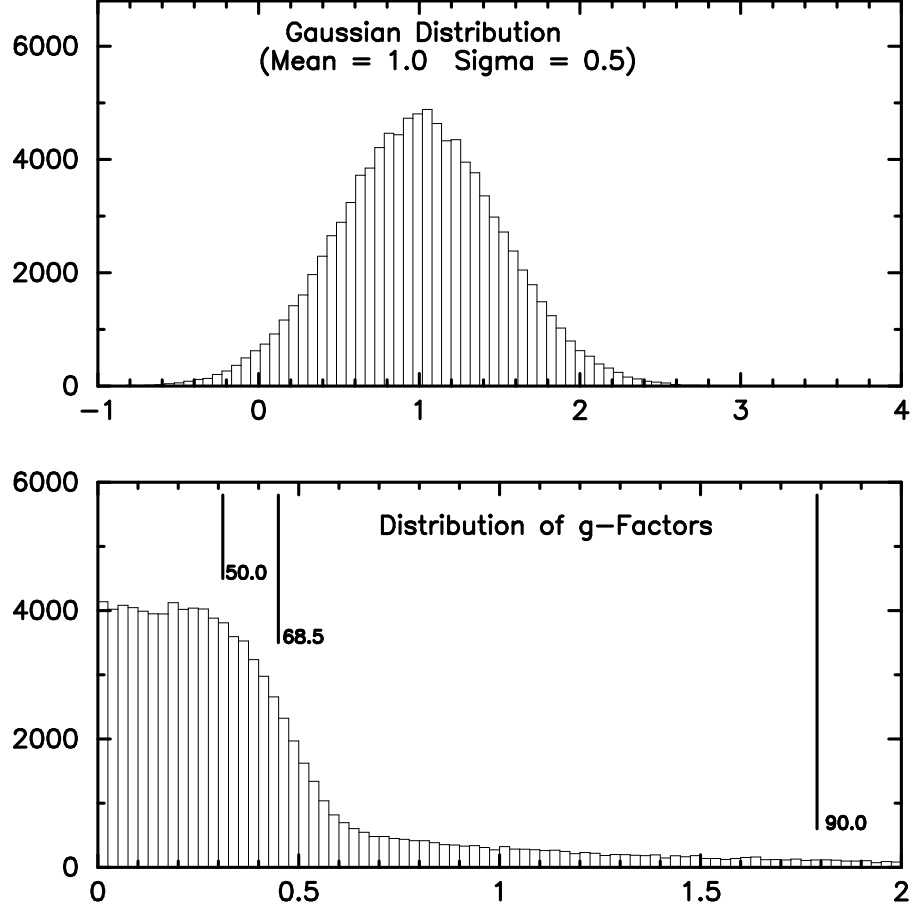


Fig. 3.— – Gaussian ($\sigma = 0.5$) Prior. The upper panel shows the input distribution of points having a Gaussian distribution with a mean of 1 and a sigma of 0.5. The lower panel shows the cumulative distribution of the multiplicative g -factors (as a function of the g -factor itself) needed to convert random samples taken from the upper panels into the mean. The g -factors required to give 50%, 68.5% (one sigma) and 90.0% (two sigma) confidence intervals for this underlying (normally distributed) sample are given as vertical bars and labeled accordingly

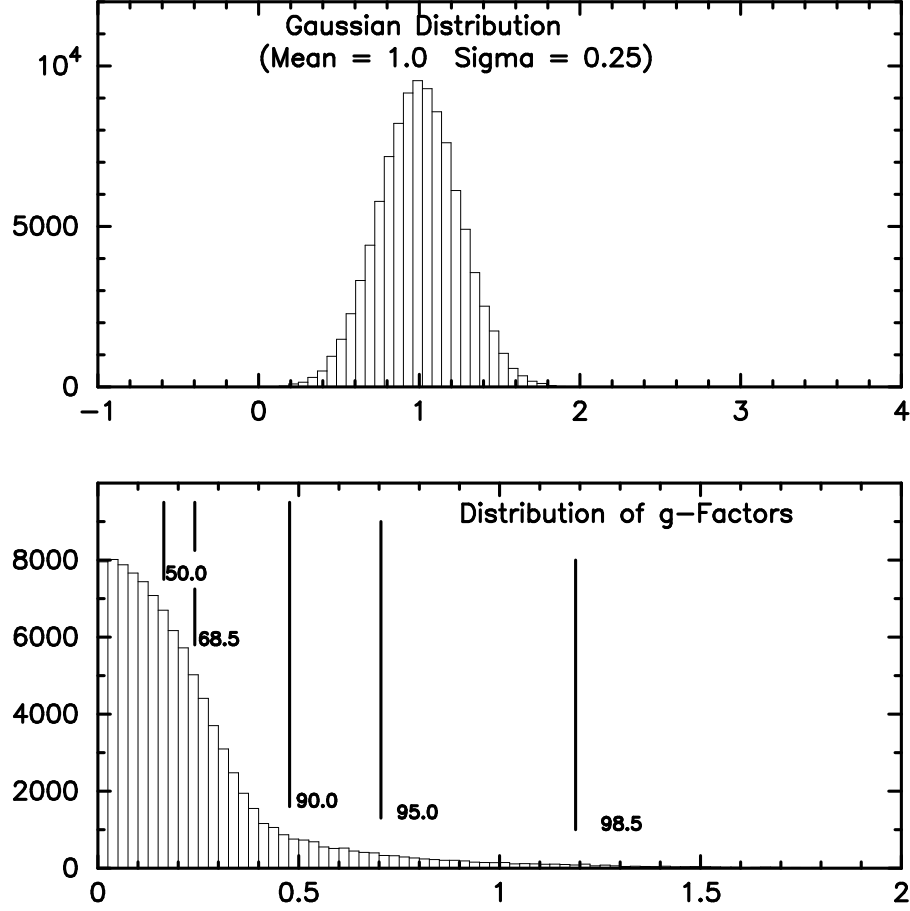


Fig. 4.— — Gaussian ($\sigma = 0.25$) Prior. The upper panel shows the input distribution of points having a Gaussian distribution with a mean of 1 and a sigma of 0.25. The lower panel shows the cumulative distribution of the multiplicative *g-factors* (as a function of the *g-factor* itself) needed to convert random samples taken from the upper panels into the mean. The *g-factors* required to give 50%, 68.5% (one sigma), 90.0% (two sigma) and 98.5% (three sigma) confidence intervals for this underlying (normally distributed) sample are given as vertical bars and labelled accordingly

Table 1. Confidence Intervals and g-Factors for Selected Priors

Confidence Interval	Uniform g-Factor	Exponential g-Factor	Gaussian(0.50) g-Factor	Gaussian(0.25) g-Factor
50.0%	± 0.414	± 0.67	± 0.31	± 0.16
68.5%	± 0.563	± 1.60	± 0.45	± 0.24
90.0%	± 4.000	± 8.5	± 1.78	± 0.48
95.0%	± 9.000	$\pm 19.$	± 4.0	± 0.70
98.5%	± 32.333	$\pm 65.$	$\pm 14.$	± 1.2